Cauchy Theorem: more general form

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Chains and cycles.

Let $(\gamma_j)_{j=1}^n$ be a finite collection of arcs (piecewise differentiable).

$(\alpha_j)_{j=1}^n$ - a finite collection of complex numbers.

$\gamma = \alpha_1\gamma_1 + \ldots + \alpha_n\gamma_n$ - a formal sum.

$$\oint_\gamma f(z)dz \overset{def}{:=} \alpha_1 \oint_{\gamma_1} f(z)dz + \ldots + \alpha_n \oint_{\gamma_n} f(z)dz$$

Def. Two chains are equivalent if they can be obtained one from another by a sequence of the following operations:

1) Permutation of two arcs, $\alpha_1\gamma_1 + \alpha_2\gamma_2 = \alpha_2\gamma_2 + \alpha_1\gamma_1$.
2) Subdivision of an arc
3) Fusion of two arcs with the same coefficients and matching endpoints to form a single arc.
4) Reparametrization of an arc.
5) Cancellation of opposite arcs: $\alpha\gamma_1 + \beta(-\gamma_1) = (\alpha-\beta)\gamma$.

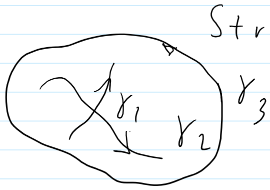
The value of $\oint_\gamma f(z)dz$ does not change after these operations.

Each chain can be written as $\gamma = \alpha_1\gamma_1 + \ldots + \alpha_n\gamma_n$ where all γ_n are different.

There is also a null-chain $\gamma = 0$.

Remark. We also denote by γ the union $\bigcup_{j=1}^n \gamma_j$.

Strictly speaking, we should use different notations!



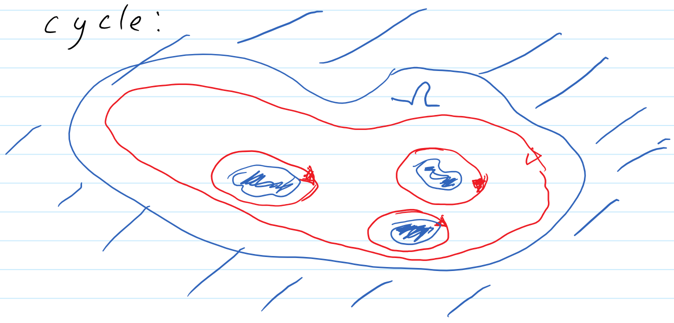
$i\gamma_1 - (2+i)\gamma_2 + 3i\gamma_3$.

Def. A chain is called a cycle if all γ_j can be chosen to be closed.

$\gamma = \alpha_1\gamma_1 + \ldots + \alpha_n\gamma_n$, where $\gamma_1, \ldots, \gamma_n$ - closed.

Linear combination of two cycles is again a cycle.

My favorite cycle:



<u>Def.</u> Let γ be a cycle, $\gamma = \alpha_1\gamma_1 + \dots + \alpha_n\gamma_n$, $z \notin \bigcup_{j=1}^{n} \gamma_j$

The <u>winding number</u> or <u>index</u> of the cycle γ:

$$n(\gamma, z) := \sum_{j=1}^{n} \alpha_j \, n(\gamma_j, z) = \sum_{j=1}^{n} \frac{\alpha_j}{2\pi i} \oint_{\gamma_j} \frac{dw}{w-z}$$

<u>Remark.</u> $P\,dx + Q\,dy$ is exact in Ω if and only if for any cycle γ: $\oint_\gamma P\,dx + Q\,dy = 0$

In complex terms: f has antiderivative iff $\forall$ cycle γ $\oint_\gamma f(z)\,dz = 0$.

<u>Def.</u> A region $\Omega \subset \mathbb{C}$ is called <u>simply-connected</u> if $\hat{\mathbb{C}} \setminus \Omega$ - connected.

<u>Remark</u> (Important!) $\hat{\mathbb{C}}$, not $\mathbb{C}$!

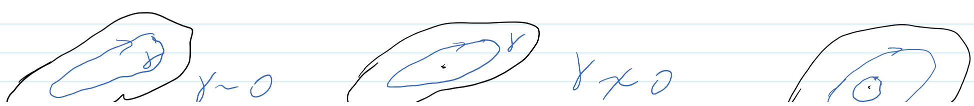
$\Omega = \mathbb{C} \setminus \{0\}$ $\quad \mathbb{C} \setminus \Omega = \{0\}$ - connected $\quad \hat{\mathbb{C}} \setminus \Omega = \{0, \infty\}$ - not connected.

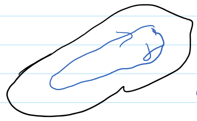
<u>Def</u> Let Ω be a region, $\gamma \subset \Omega$ - a chain.

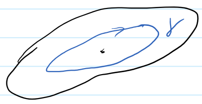
We say that γ is <u>homologous to 0 with respect to Ω</u> if for any $z \notin \Omega$, $n(\gamma, z) = 0$.

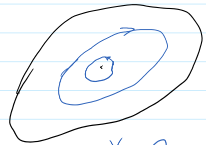
<u>Notation:</u> $\gamma \sim 0$.

Heuristically: γ does not wind around points outside of Ω.




$\gamma \sim 0$


$\gamma \not\sim 0$


$\gamma \sim 0$

Def. $\gamma_1 \sim \gamma_2$ if $\gamma_1 - \gamma_2 \sim 0$.

Observation. If Ω is simply connected then for <u>any</u> cycle $\gamma \subset \Omega$, $\gamma \sim 0$.

Proof. Let $z \notin \Omega$. Then, since $\hat{\mathbb{C}} \setminus \Omega$ is connected, it belongs to the unbounded component of $\mathbb{C} \setminus \Omega$ (the only one), which is subset of unbounded component of $\mathbb{C} \setminus \gamma$. So $n(\gamma, z) = 0$

Remark. As proved in Ahlfors: opposite is also true:
$(\forall \gamma \subset \Omega - \text{cycle},\ \gamma \sim 0) \Rightarrow \Omega$ is simply connected.

Theorem (General Cauchy Theorem).
Let $f \in A(\Omega)$, $\gamma \sim 0$ wrt Ω.
Then $\oint_\gamma f(z)\,dz = 0$

Corollary. $f \in A(\Omega)$, $\gamma_1 \sim \gamma_2 \Rightarrow \oint_{\gamma_1} f(z)\,dz = \oint_{\gamma_2} f(z)\,dz$

We will prove a global version of Cauchy Integral Formula:

Theorem (General Cauchy Integral Formula)
Let $f \in A(\Omega)$, $\gamma \sim 0$ wrt. Ω
Then $\forall z \notin \gamma$,

$n(\gamma) f(z) = \frac{1}{\ } \oint f(\zeta)\ \ d\zeta$

$$n(\gamma,z)f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof that CIF $\Rightarrow$ Cauchy.

Consider $F(z) := f(z)(z-z_0)$ for some $z_0 \in \Omega \setminus \gamma$.

Then $\frac{1}{2\pi i}\oint_\gamma f(z)\,dz = \frac{1}{2\pi i}\oint_\gamma \frac{F(z)}{z-z_0}\,dz = n(\gamma,z_0)F(z_0) = 0$ ∎

Proof of CIF:

Lemma Let $g \in A(\Omega)$, $z_0 \in \Omega$.

Then $\lim_{\substack{(z,\zeta)\to(z_0,z_0)\\ z\neq\zeta}} \frac{g(z)-g(\zeta)}{z-\zeta} = g'(z_0)$

Proof. Need: $\forall \varepsilon > 0\ \exists \delta > 0: \sqrt{|z-z_0|^2 + |\zeta - z_0|^2} < \delta \Rightarrow \left|\frac{g(z)-g(\zeta)}{z-\zeta} - g'(z_0)\right| < \varepsilon.$

Know: $\exists \delta > 0 : |w - z_0| < \delta \Rightarrow |g'(w) - g'(z_0)| < \varepsilon.$

Let $\gamma = [\zeta, z]$ - the interval from ζ to z.

Then $g'(z_0) = \oint_\gamma \frac{g'(z_0)}{z-\zeta}\,dw \quad \left(\oint_\gamma dw = z - \zeta\right)$

$$\frac{g(z)-g(\zeta)}{z-\zeta} = \oint_\gamma \frac{g'(w)}{z-\zeta}\,dw$$

So if $\begin{array}{l}|z-z_0| < \delta\\ |\zeta - z_0| < \delta\end{array}$ then $\forall w \in \gamma$, $|w - z_0| < \delta \Rightarrow |g'(w) - g'(z_0)| < \varepsilon$.

So $\left|\frac{g(z)-g(\zeta)}{z-\zeta} - g'(z_0)\right| = \left|\oint_\gamma \frac{g'(w)-g'(z_0)}{z-\zeta}\,dw\right| < \frac{\varepsilon}{|z-\zeta|}\cdot \ell(\gamma) = \varepsilon$ ∎

Let now $F(z,\zeta) := \begin{cases} \frac{f(z)-f(\zeta)}{z-\zeta}, & z \neq \zeta \\ f'(z), & z = \zeta.\end{cases}$

Observe: 1) F is a continuous function.

Indeed: (z,ζ) $z \neq \zeta$ – continuity at (z,ζ) obvious.

(z_0,z_0): By Lemma, $\lim_{\substack{(z,\zeta)\to(z_0,z_0)\\ z\neq\zeta}} F(z,\zeta) = f'(z_0) = F(z_0,z_0)$

$\lim_{\substack{(z,\zeta)\to(z_0,z_0)\\ z=\zeta}} F(z,\zeta) = \lim_{z\to z_0} f'(z) = f'(z_0)$, continuity of derivative!

2) $F(z,\zeta) = F(\zeta, z)$

3) For each ζ_0, $z \to F(z,\zeta_0)$ is analytic in Ω.

Indeed $F(z,\zeta_0) = \frac{f(z)-f(\zeta_0)}{z-\zeta_0}$ is analytic for $z \neq \zeta_0$.

$$\lim_{z\to\zeta_0} F(z,\zeta_0)(z-\zeta_0) = \lim_{z\to\zeta_0}(f(z)-f(\zeta_0)) = 0.$$

So the singularity at ζ_0 is removable, and $F(\cdot,\zeta_0)$ is analytic in Ω.

Define now: $\Omega' = \{z \in \mathbb{C}\setminus\gamma : n(\gamma,z)=0\}$.

$\boxed{\mathbb{C}\setminus\Omega \subset \Omega'}$ (because $\gamma\sim 0$).

Define
$$h(z) = \begin{cases} \frac{1}{2\pi i}\oint_\gamma F(z,\zeta)\,d\zeta, & z\in\Omega \\ \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta, & z\in\Omega' \end{cases}$$

For $z \in \Omega'\cap\Omega$, $\frac{1}{2\pi i}\oint_\gamma F(z,\zeta)d\zeta = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta - \underbrace{\frac{f(z)}{2\pi i}\oint_\gamma\frac{d\zeta}{\zeta-z}}_{f(z)n(\gamma,z)} = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{z-\zeta}d\zeta.$

So h is well-defined.

$h\in A(\Omega')$ (it is a Cauchy integral of f).

For $z\in\Omega\setminus\gamma$, $h(z) = \underbrace{\frac{1}{2\pi i}\oint_\gamma\frac{f(\zeta)}{\zeta-z}d\zeta}_{\text{Cauchy integral}} - \underbrace{n(\gamma,z)f(z)}_{\text{analytic}} \in A(\Omega\setminus\gamma)$.

So $h\in A(\mathbb{C}\setminus\gamma)$.

Claim. $\forall z_0\in\gamma$, h is analytic at z_0.

Proof (of Claim).

Consider $B(z_0,r)\subset\Omega$. Let Γ be any closed curve in $B(z_0,r)$.

Then $\oint_\Gamma h(z)\,dz = \frac{1}{2\pi i}\oint_\Gamma\left(\oint_\gamma F(z,\zeta)\,d\zeta\right)dz \overset{\text{continuous}}{=} \oint_\gamma\left(\oint_\Gamma F(z,\zeta)dz\right)d\zeta.$

But $\forall\zeta$ $z\to F(z,\zeta)\in A(\Omega) \Rightarrow \oint_\Gamma F(z,\zeta)dz = 0.$

So $\forall\,\Gamma$ - closed, $\Gamma\subset B(z_0,r)$ we have $\oint_\Gamma h(z)dz \equiv 0.$

So, by Morera, $h\in A(B(z_0,r))$ ∎

Thus $h\in A(\mathbb{C})$.

Also $\lim_{|z|\to\infty} h(z) = \lim_{|z|\to\infty}\left(\int\frac{f(\zeta)}{z-\zeta}d\zeta\right) = 0.$

So, by maximum principle, $h\equiv 0$

So, for $z\in\Omega$

$$0 = h(z) = \frac{1}{2\pi i}\oint_\gamma\frac{f(\zeta)}{\zeta-z}d\zeta - n(\gamma,z)f(z) \blacksquare$$

Corollary. If Ω is simply connected, then $\forall f \in A(\Omega)$, $z_0 \in \Omega$, $\gamma \subset \Omega$ - cycle, $z_0 \notin \gamma$:

1) $\oint\limits_\gamma f(z)\,dz = 0$

2) $f(z_0)\,n(\gamma, z_0) = \frac{1}{2\pi i} \oint\limits_\gamma \frac{f(z)}{z - z_0}\,dz$

Proof. $\gamma \sim 0$

Corollary Let Ω be simply connected. $f \in A(\Omega)$. $\forall z \in \Omega$ $f(z) \neq 0$.

Then $\exists\ g \in A(\Omega) : e^g = f$ (branch of logarithm)

$\forall n \in \mathbb{N}\ \exists h \in A(\Omega)$: $h^n = f$ (branch of n-th root).

Proof. Note that $\frac{f'(z)}{f(z)} \in A(\Omega)$.

Thus $\exists \tilde{g}$: $\tilde{g}'(z) = \frac{f'(z)}{f(z)}$, $\tilde{g} \in A(\Omega)$ (antiderivative).

Fix $z_0 \in \Omega$. Take $g(z) := \tilde{g}(z) - \tilde{g}(z_0) + \operatorname{Log} f(z_0)$

Then 1) $e^{g(z_0)} = e^{\operatorname{Log} f(z_0)} = f(z_0)$

2) $\left(f(z)e^{-g(z)}\right)' = f'(z)e^{-g(z)} - f(z) \cdot \underbrace{g'(z)}_{\frac{f'(z)}{f(z)}} e^{-g(z)} = 0$

Thus $f(z)e^{-g(z)} \equiv const = f(z_0)e^{-g(z_0)} = 1 \Rightarrow$

$f(z) = e^{g(z)}$

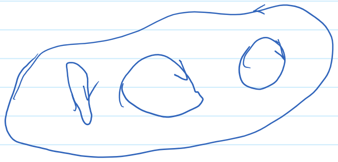
Now take $h(z) := \exp\left(\frac{g(z)}{n}\right)$

Oriented boundary.

Let $\gamma = \gamma_1 + \ldots + \gamma_n$ be a cycle, each of the γ_i is a simple closed arc. We say that γ is an oriented boundary of region Ω (or γ bounds Ω) if

1) $\forall z \in \Omega$, $n(\gamma, z) = 1$

2) $\forall z \in (\Omega \cup \gamma)^c$, $n(\gamma, z) = 0$



Corollary If γ bounds Ω and $f \in A(\Omega \cup \gamma)$, then

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0}\, dz = \begin{cases} f(z_0), & z_0 \in \Omega \\ 0, & z_0 \notin \Omega \\ & z_0 \notin \gamma \end{cases}$$